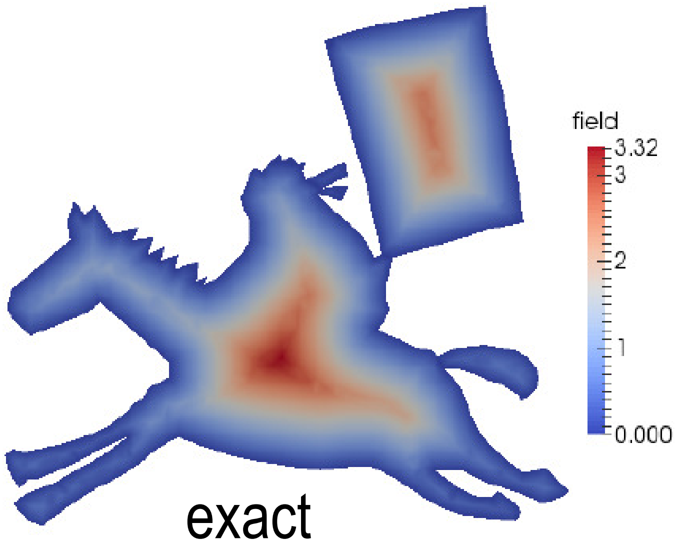
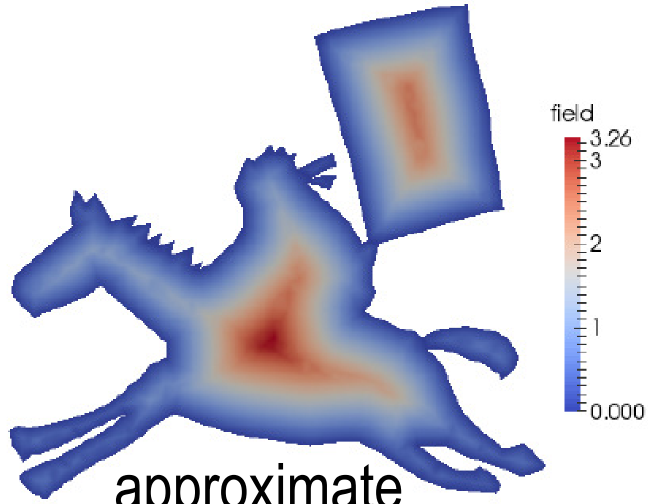
Variational Methods for Accurate Distance Function Estimation



exact



approximate

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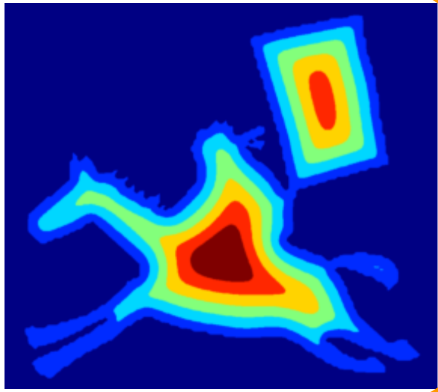
Distance function $d(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \partial\Omega) = \min_{y \in \partial\Omega} \|\boldsymbol{x} - \boldsymbol{y}\|$

distance to
a curve
(surface)
$d(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \partial\Omega)$



$$d(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \partial\Omega)$$

$$d = 0 \text{ on } \partial\Omega$$

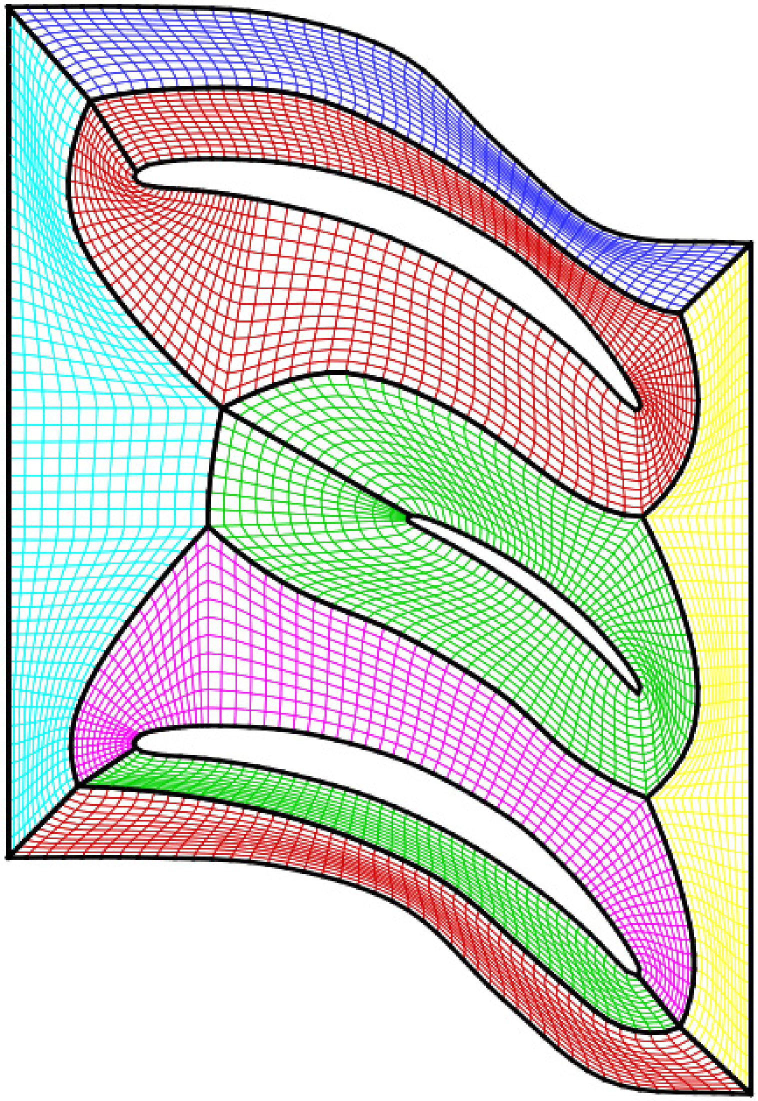
$$\partial^k d / \partial \boldsymbol{n}^k = \delta_{1k} \text{ on } \partial\Omega$$

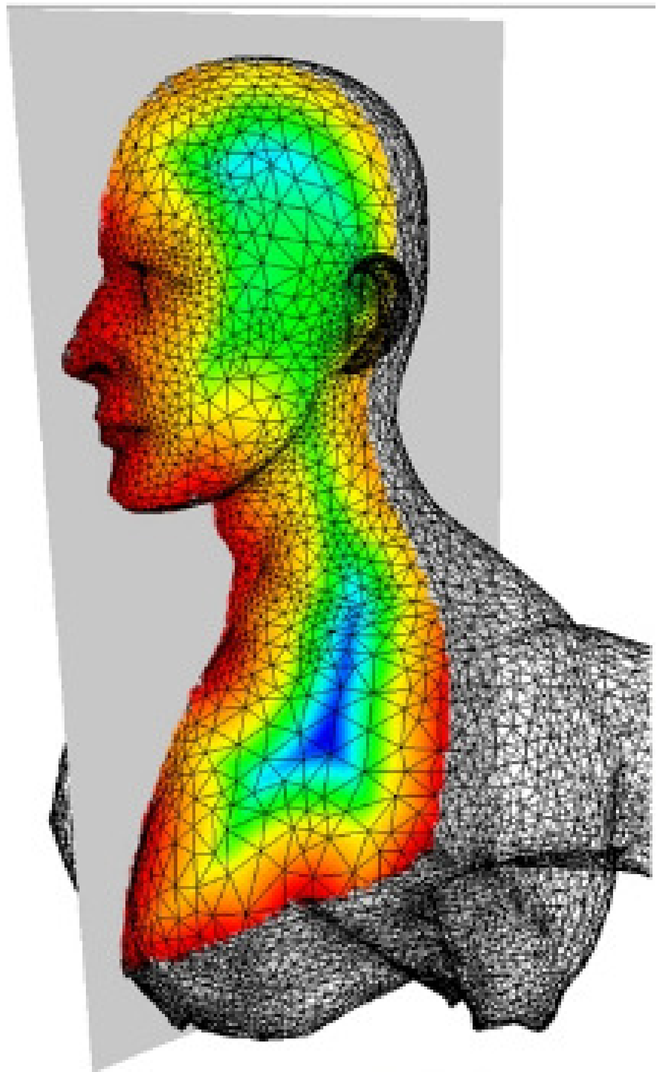
$$k = 1, 2, \ldots$$

$$\underbrace{|\nabla d(\boldsymbol{x})| = 1}$$ eikonal equation

It is not easy to solve it numerically with high accuracy: it is non-linear and the solution develops singularities

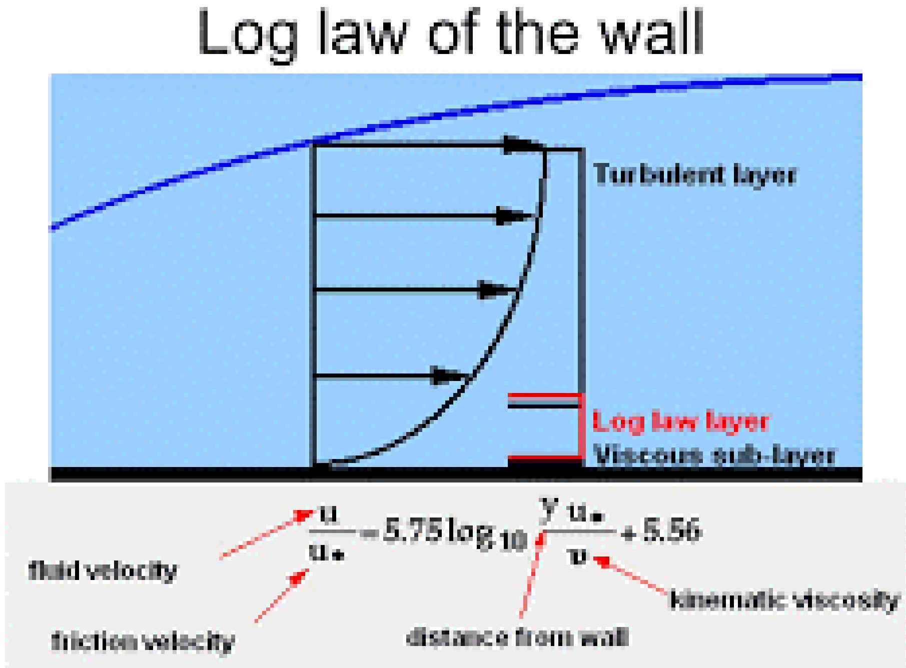
Distance function for mesh generation





The law of the wall in fluid dynamics

The **law of the wall** states that the average velocity of a turbulent flow at a certain point is proportional to the logarithm of the distance from that point to the "wall", or the boundary of the fluid region.



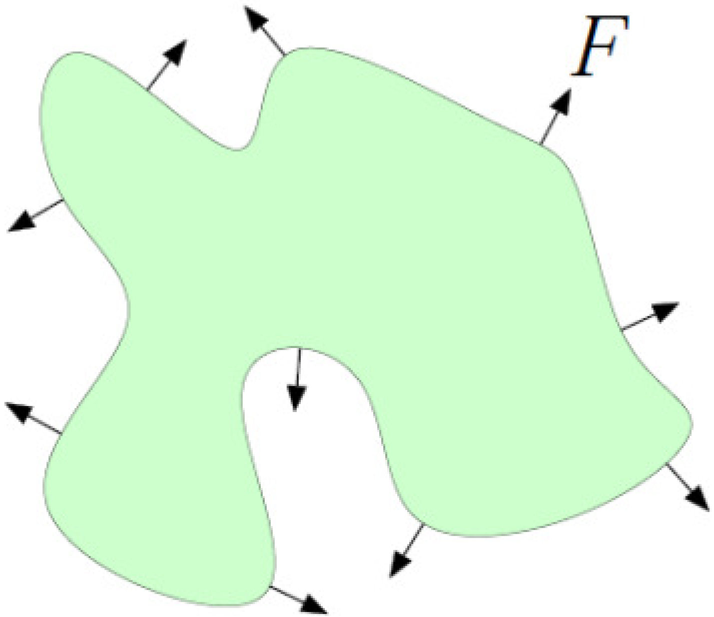
Surface reconstruction from scattered point data

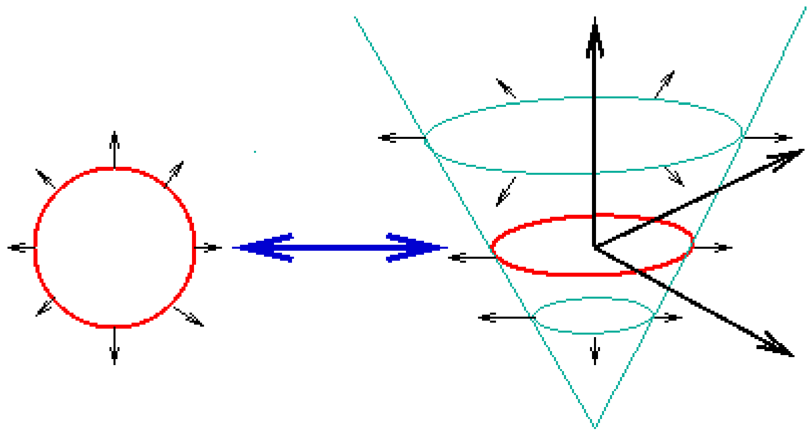
Fatih Calakli and Gabriel Taubin
SSD: Smooth Signed Distance Surface Reconstruction
Computer Graphics Forum Vol. 30, No. 7, 2011.

Level sets and re-distancing

Evolving Curves and Surfaces:

- Propagate curve according to speed function $v = F\mathbf{n}$
- F depends on space, time, and the curve itself
- Surfaces in three dimensions





Level set approach: Represent the evolving curve as the zero level set of a function $\varphi(x,y,t)$

Level set equation →

$$\frac{\partial \varphi}{\partial t} = -F\left|\nabla \varphi\right|$$

Level sets and re-distancing: area of active research

ELSEVIER

Journal of Computational Physics

Volume 330, 1 February 2017, Pages 268-281

Revisiting the redistancing problem using the Hopf–Lax formula

Byungjoon Lee [a], Jérôme Darbon [c], Stanley Osher [b], Myungjoo Kang [a]

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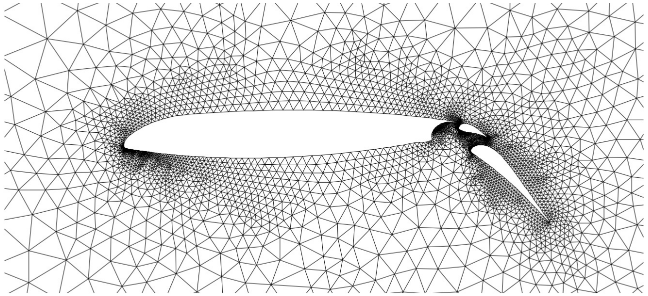
Journal of Computational Physics

Volume 365, 15 July 2018, Pages 7-17

Parallel redistancing using the Hopf–Lax formula

Michael Royston [a], Andre Pradhana [a], Byungjoon Lee [b], Yat Tin Chow [a], Wotao Yin [a], Joseph Teran [a], Stanley Osher [a]

Applications of distance functions in Computational Maths & Physics



$$\begin{cases} L[u(x)] = f(x) & \text{in } \Omega \\ u(x) = \varphi(x) & \text{on } \partial\Omega \end{cases}$$

A boundary value problem:

← Dirichlet boundary condition

The characteristic function method of Leonid Kantorovich:

Looking for an approximation of $u(x)$ in the form

$$u(x) = \varphi(x) + \omega(x)\Phi(x), \quad \Phi(x) = \sum_{i=1}^{n} c_i e_i(x)$$

$$\omega(x) = 0 \text{ on } \partial\Omega, \quad \omega(x) > 0 \text{ in } \Omega, \quad |\nabla\omega(x)| \geq \alpha > 0 \text{ on } \partial\Omega,$$

Then We have a lot of freedom in choosing basis $\{e_i(x)\}$ functions, as they don't need to vanish on $\partial\Omega$

Extended to other types of boundary conditions (Rvachev)

Rvachev's extension of the characteristic function method of Kantorovich

Type of Boundary Condition	Mathematical Formulation	Corresponding Solution Structure
Dirichlet	$u_{\|\partial\Omega} = \varphi$	$u = \omega\Phi + \varphi$
Neumann	$\frac{\partial u}{\partial n}_{\|\partial\Omega} = \varphi$	$u = \Phi - \omega D_1^{\omega}(\Phi) + \omega\varphi + \omega^2\Phi$
3-rd kind	$(\frac{\partial u}{\partial n} + hu)_{\|\partial\Omega} = \varphi$	$u = \Phi - \omega D_1^{\omega}(\Phi) - h\omega\Phi + \omega\varphi + \omega^2\Phi$
Mixed	$u_{\|\partial\Omega_1} = \varphi$ $(\frac{\partial u}{\partial n} + hu)_{\|\partial\Omega_2} = \psi$	$u = \omega_1\Phi + \frac{\omega_1\omega_2}{\omega_1+\omega_2}(\psi + \omega_2\Phi - D_1^{\omega_2}(\omega_1\Phi) - D_1^{\omega_2}(\varphi) - h\omega_1\Phi - h\varphi) + \varphi$

$$\omega(x) = 0 \text{ on } \partial\Omega, \quad \omega(x) > 0 \text{ in } \Omega$$

$$\partial\omega/\partial\boldsymbol{n} = 1 \text{ on } \partial\Omega$$

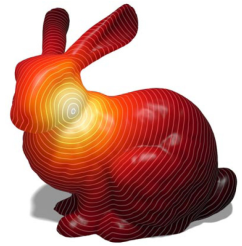
$$\partial^k\omega/\partial\boldsymbol{n}^k = 0 \text{ on } \partial\Omega, \; k = 2,3,\ldots$$

So distance function approximations which are accurate near the boundary are needed

Variational and PDEs methods for estimating $\mathbf{dist}(x,\partial\Omega)$

$$d(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x},\partial\Omega): \quad |\nabla d(\boldsymbol{x})| = 1 \text{ in } \Omega \quad d(\boldsymbol{x}) = 0 \text{ on } \partial\Omega$$

Geodesics in Heat



KEENAN CRANE
Caltech
CLARISSE WEISCHEDEL, MAX WARDETZKY
University of Göttingen

$$(\mathrm{id} - t\Delta)v_t = 0 \quad \text{on} \quad M\backslash\gamma$$
$$v_t = 1 \quad \text{on} \quad \gamma\,.$$

$$\lim_{t\to 0} -\frac{\sqrt{t}}{2}\log v_t = \phi$$

$$\Delta_p u \equiv \mathrm{div}\left(|\nabla u|^{p-2}|\nabla u|\right) = -1 \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega \quad u(\boldsymbol{x}) \to d(\boldsymbol{x}) \text{ as } p \to \infty$$

$$\Delta u = -1 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad v(\boldsymbol{x}) = \frac{2u}{|\nabla u| + \sqrt{|\nabla u|^2 + 2u}}$$
$$v = 0 \quad \text{and} \quad \partial v/\partial \boldsymbol{n} = 1 \quad \text{on} \quad \partial\Omega$$

P.R.Spalding
P.G.Tucker (Cambridge)

$$\int_\Omega \left(|\nabla u(\boldsymbol{x})| - 1\right)^2 d\boldsymbol{x} \to \min \qquad \Delta u = \underbrace{\mathrm{div}\left(\nabla u / |\nabla u|\right)}_{\text{level-set curvature}}$$

Euler-Lagrange equation

Geodesics in Heat of K. Crane, C. Weischedel , M. Wardetzky

$$v - t\Delta v = 0 \ \text{in} \ \ \Omega, \qquad v = 1 \ \text{on} \ \ \partial\Omega, \qquad t > 0$$

Screened Poisson : Linear PDE – easy to solve numerically

Substitution : $v(x) = \exp\left\{-u(x)/\sqrt{t}\right\}$ **Hopf-Cole transformation**

$$\frac{\partial v}{\partial x_i} = -\frac{v}{\sqrt{t}}\frac{\partial u}{\partial x_i}, \quad \frac{\partial^2 v}{\partial x_i^2} = \frac{v}{t}\left|\frac{\partial u}{\partial x_i}\right|^2 - \frac{v}{\sqrt{t}}\frac{\partial^2 u}{\partial x_i^2}$$

$$0 = v - t\Delta v = v\left[\left(1 - |\nabla u|^2\right) + \sqrt{t}\Delta u\right], \quad u = 0 \ \text{on} \ \ \partial\Omega$$

$$\left(1 - |\nabla u|^2\right) + \sqrt{t}\Delta u = 0 \ \text{approximates} \ |\nabla u|^2 = 1, \quad u = 0 \ \text{on} \ \ \partial\Omega$$

eikonal equation

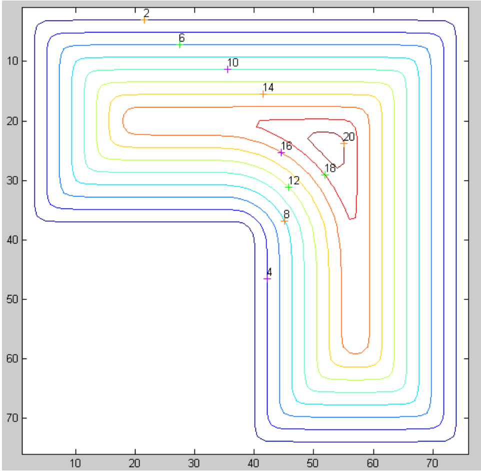
Geodesics in Heat of K. Crane, C. Weischedel , M. Wardetzky

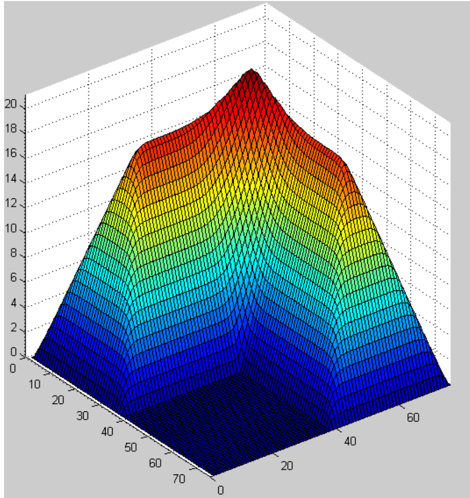
$$\left(1-|\nabla u|^2\right)+\sqrt{t}\,\Delta u = 0, \quad |\nabla u|^2 \approx 1 \text{ if } t << 1$$

Substitution: $\quad v(x) = \exp\left\{-u(\boldsymbol{x})/\sqrt{t}\right\}$

$$v - t\,\Delta v = 0 \text{ in } \Omega, \qquad v = 1 \text{ on } \partial\Omega$$

Linear PDE – easy to solve numerically





Substitution: $\quad v(\boldsymbol{x}) = 1 - w(x)$

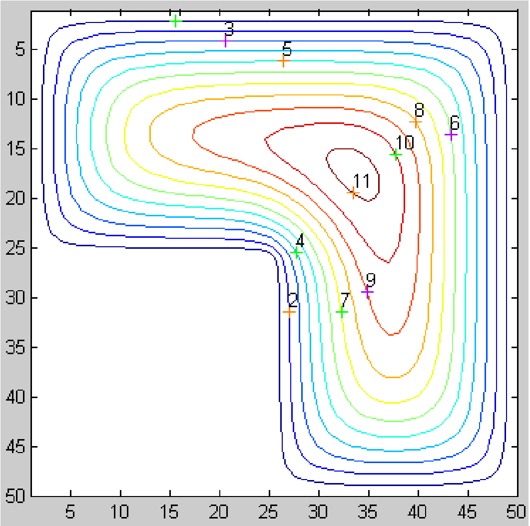
$$w - t\,\Delta w = 1 \text{ in } \Omega, \qquad w = 0 \text{ on } \partial\Omega$$

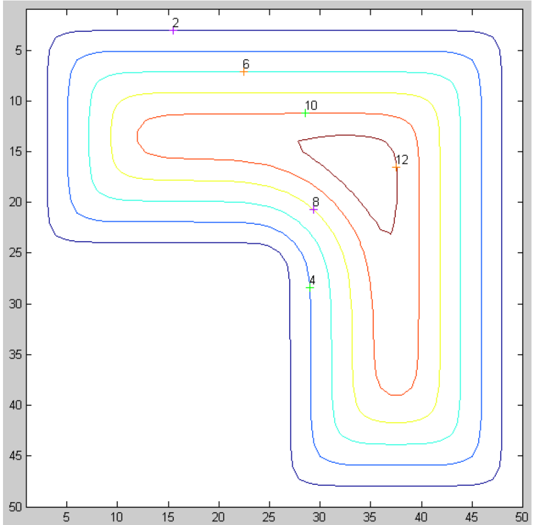
$$u(\boldsymbol{x}) = -\sqrt{t}\,\ln\left[1 - w(\boldsymbol{x})\right]$$

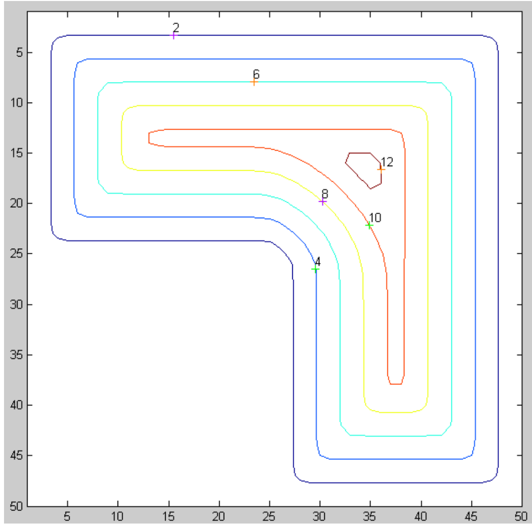
Laplacian

```
rhs = ones(N,1);
u = -sqrt(t)*log(1-(t*D+eye(N))\rhs);
```

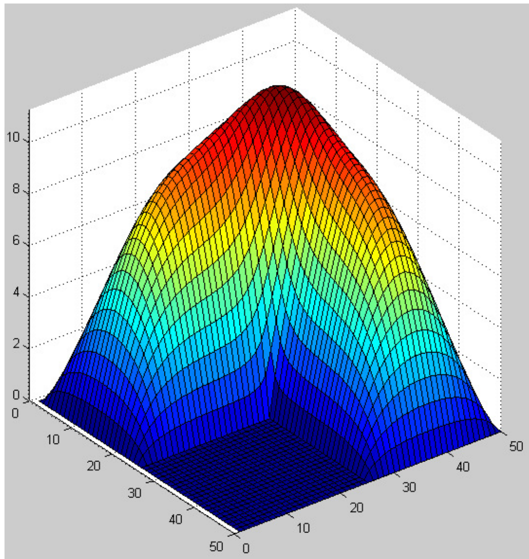
Geodesics in Heat of K. Crane, C. Weischedel , M. Wardetzky

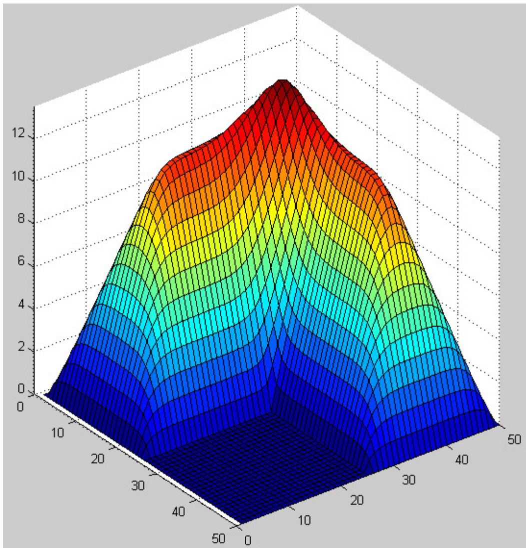


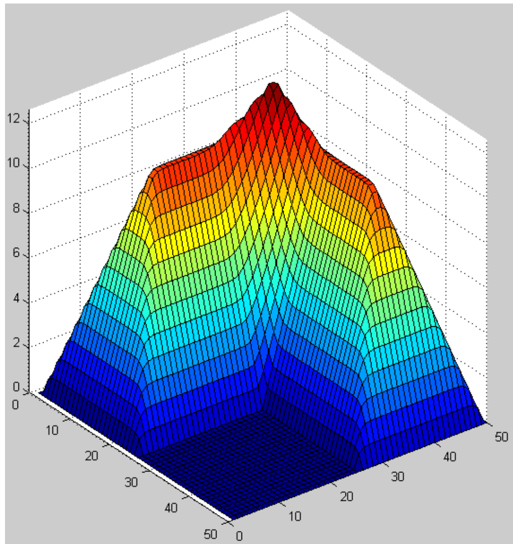




$t = 20$ $t = 2$ $t = 0.2$

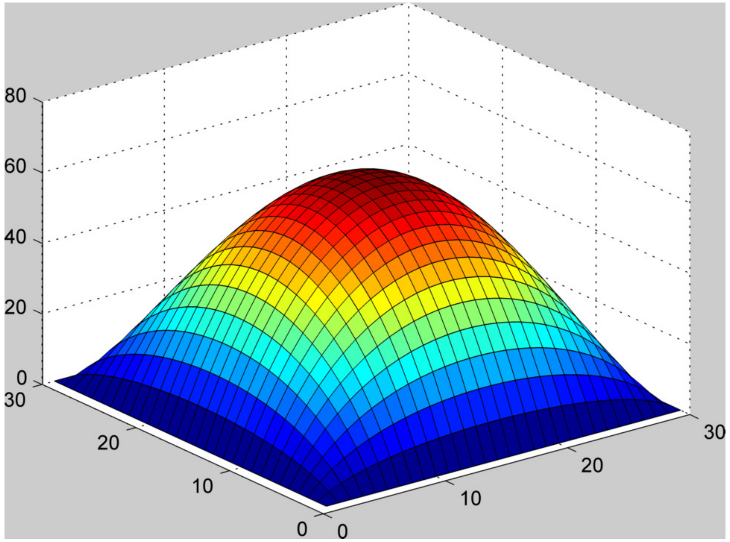




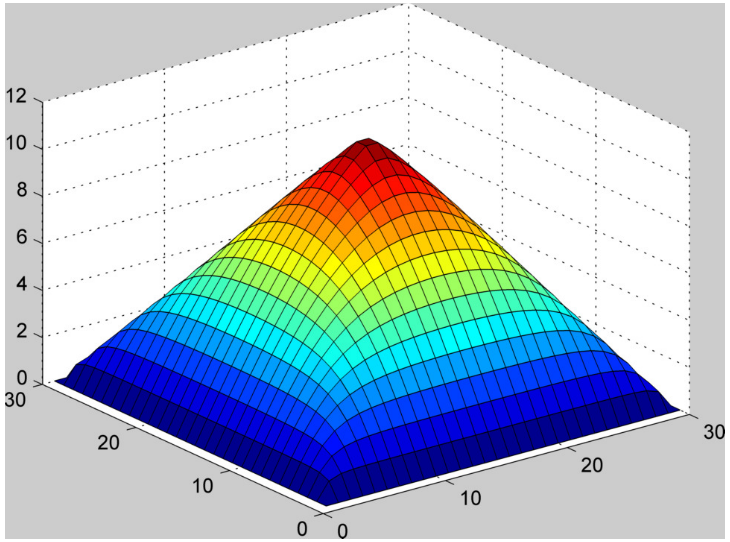


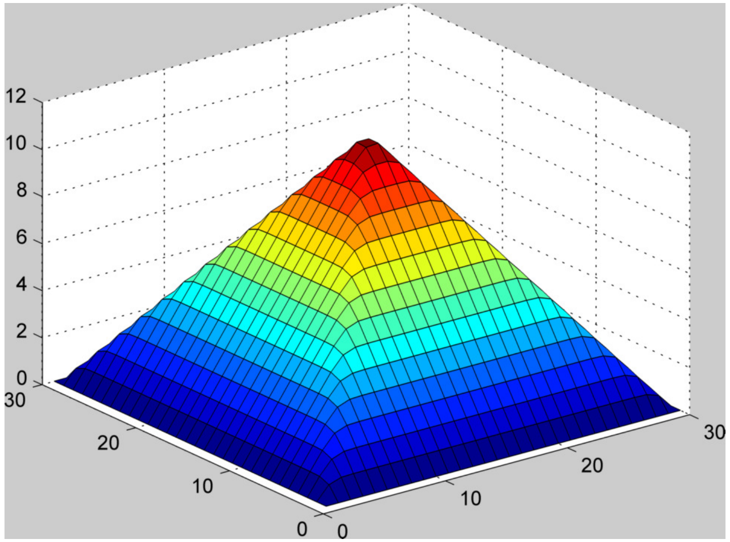
Geodesics-in-heat vs Spalding-Tucker normalization

$$\Delta u = -1 \ \text{in} \ \Omega, \quad u = 0 \ \text{on} \ \partial\Omega$$



$$v(\boldsymbol{x}) = \frac{2u}{|\nabla u| + \sqrt{|\nabla u|^2 + 2u}}$$





Geodesics-in-Heat

$$w - t\,\Delta w = 1 \ \text{in} \ \Omega, \quad w = 0 \ \text{on} \ \partial\Omega$$

$$u(\boldsymbol{x}) = -\sqrt{t}\,\ln[1 - w(\boldsymbol{x})]$$

A simple variational approach

COMPUTER GRAPHICS *forum*
Volume 34 (2015), number 8 pp. 104–118

On Variational and PDE-Based Distance Function Approximations

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fayolle@u-aizu.ac.jp

$$d(\boldsymbol{x}) = \mathrm{dist}(\boldsymbol{x}, \partial\Omega) \qquad |\nabla d(\boldsymbol{x})| = 1$$ eikonal equation

$$d = 0 \quad \text{and} \quad \partial d / \partial \boldsymbol{n} = 1 \quad \text{on} \quad \partial\Omega,$$
$$\partial^k d / \partial \boldsymbol{n}^k = 0 \quad \text{on} \quad \partial\Omega, \quad k = 2, 3, \ldots$$

$$E(u) = \int_\Omega \left(|\nabla u(\boldsymbol{x})| - 1 \right)^2 d\boldsymbol{x} \rightarrow \min \qquad u = 0 \;\; \text{on} \;\; \partial\Omega$$

A simple splitting scheme

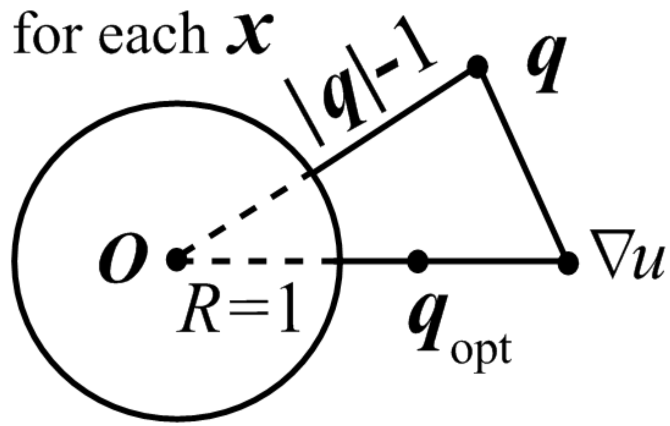
$$E(u) = \int_\Omega \left(|\nabla u(\boldsymbol{x})| - 1\right)^2 d\boldsymbol{x} \rightarrow \min \qquad \boldsymbol{q} = \nabla u$$

$$E_r(u, \boldsymbol{q}) = \int_\Omega \left\{ \left(|\boldsymbol{q}| - 1\right)^2 + r\left(\boldsymbol{q} - \nabla u\right)^2 \right\} d\boldsymbol{x} \rightarrow \min$$

Optimising w.r.t. $u(\boldsymbol{x})$: $\Delta u = \operatorname{div} \boldsymbol{q}$

Optimising w.r.t. $\boldsymbol{q}$: $\boldsymbol{q} = c(\boldsymbol{x}) \nabla u$

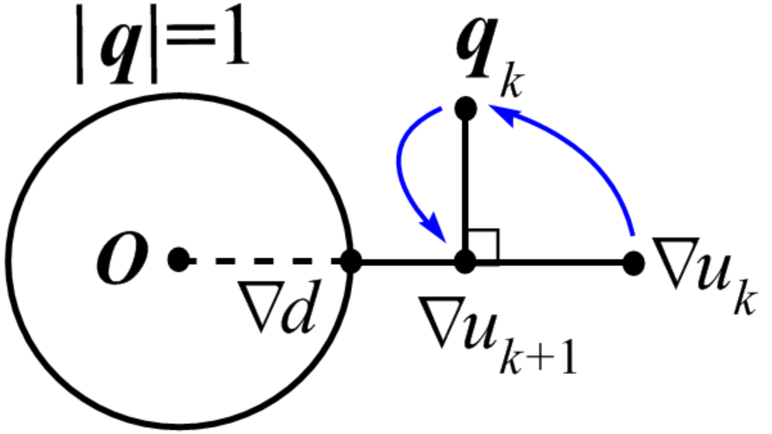
$$c = \frac{1 + r|\nabla u|}{(1 + r)|\nabla u|}$$



A simple splitting scheme

$$E_r(u,\boldsymbol{q}) = \int_\Omega \left\{ \left(|\boldsymbol{q}| - 1 \right)^2 + r\left(\boldsymbol{q} - \nabla u \right)^2 \right\} d\boldsymbol{x} \rightarrow \min$$

$$\boldsymbol{q}_k = \frac{1 + r|\nabla u_k|}{(1+r)|\nabla u_k|} \nabla u_k \qquad \Delta u_{k+1} = \operatorname{div} \boldsymbol{q}_k$$



$$\int_\Omega \left(|\boldsymbol{q}_k| - 1 \right)^2 d\boldsymbol{x} \le E_r(u_k, \boldsymbol{q}_k)$$

$$\le E_r(u_k, \nabla u_k) = \int_\Omega \left(|\nabla u_k| - 1 \right)^2 d\boldsymbol{x}$$

… and a convergence can be established

A splitting scheme

$$E_r(u,\boldsymbol{q}) = \int_{\Omega} \left\{ \left(|\boldsymbol{q}| - 1 \right)^2 + r\left(\boldsymbol{q} - \nabla u\right)^2 \right\} d\boldsymbol{x} \to \min$$

$$\boldsymbol{q}_k = \frac{1 + r|\nabla u_k|}{(1+r)|\nabla u_k|} \nabla u_k$$

$$\Delta u_{k+1} = \operatorname{div} \boldsymbol{q}_k$$

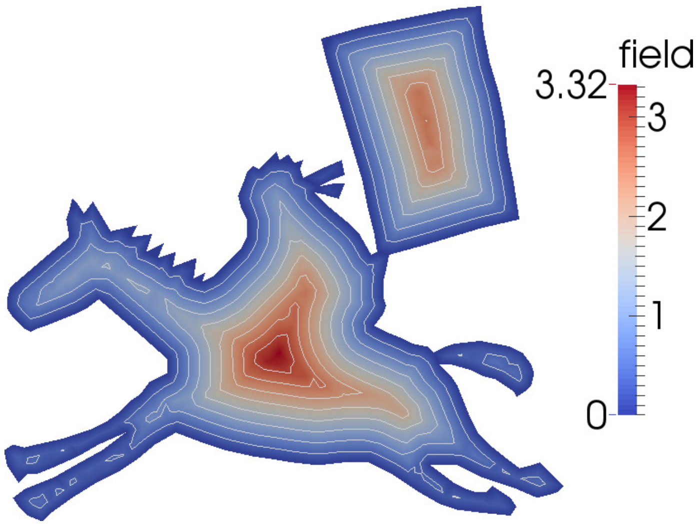
the most computationally expensive step

$Au = b$ The same system of linear equations is solved for each iteration.

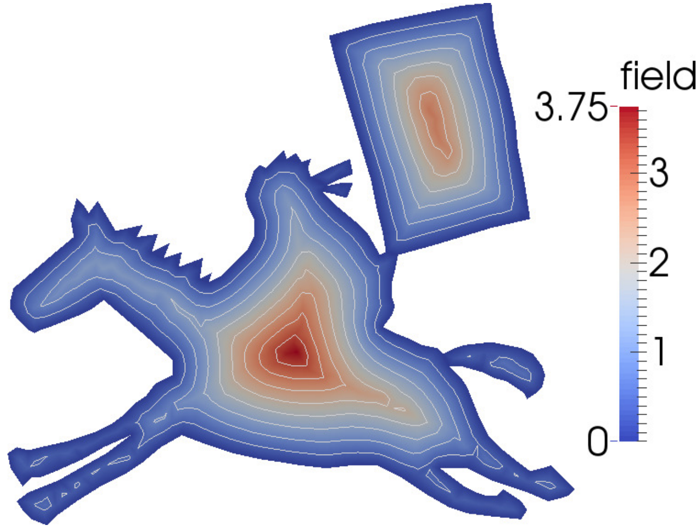
$A = L L^T$ So the Cholesky decomposition is used

Results for 2D domains

$$E_r(u,\boldsymbol{q}) = \int_{\Omega} \left\{ \left(|\boldsymbol{q}| - 1 \right)^2 + r\left(\boldsymbol{q} - \nabla u\right)^2 \right\} d\boldsymbol{x} \rightarrow \min$$



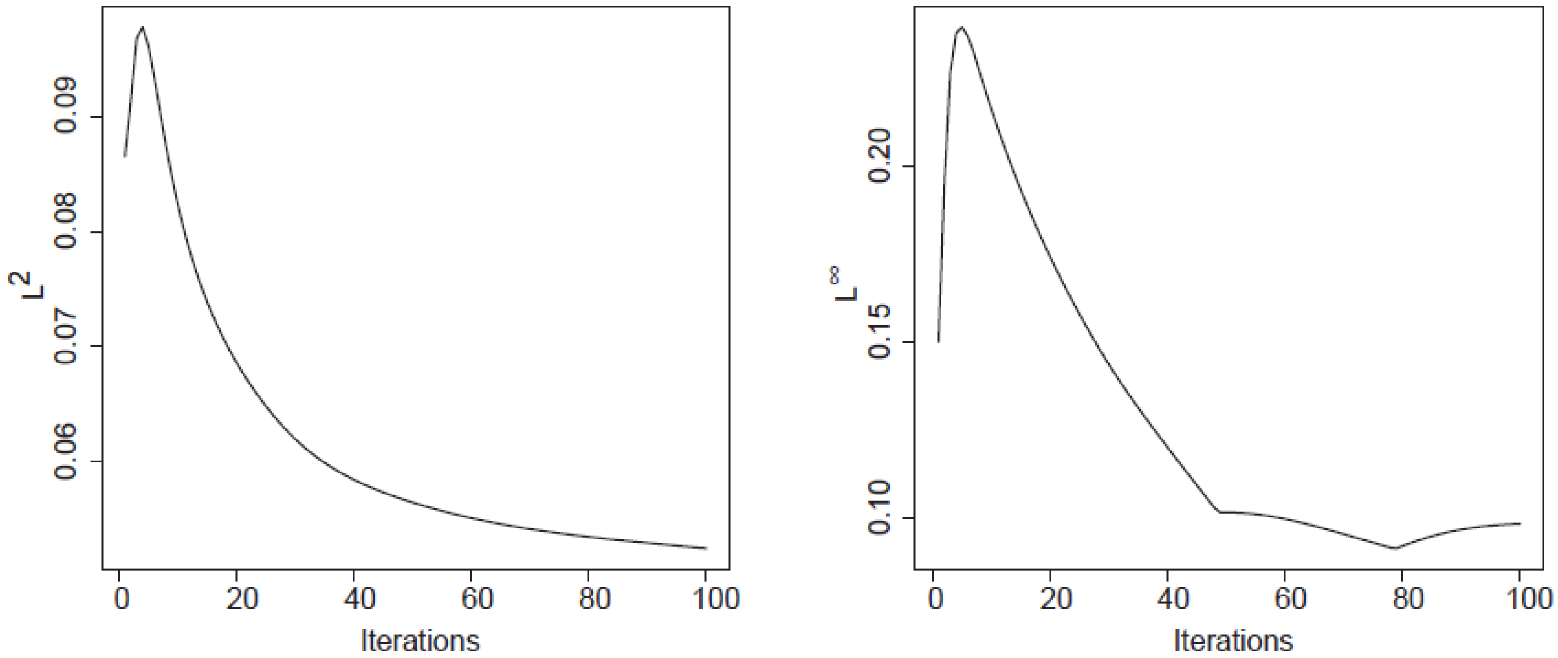
exact distance



distance by splitting scheme

3. Results: speed of convergence

$$E_r(u,\boldsymbol{q}) = \int_\Omega \left\{ \left(|\boldsymbol{q}|-1\right)^2 + r(\boldsymbol{q}-\nabla u)^2 \right\} d\boldsymbol{x} \rightarrow \min$$



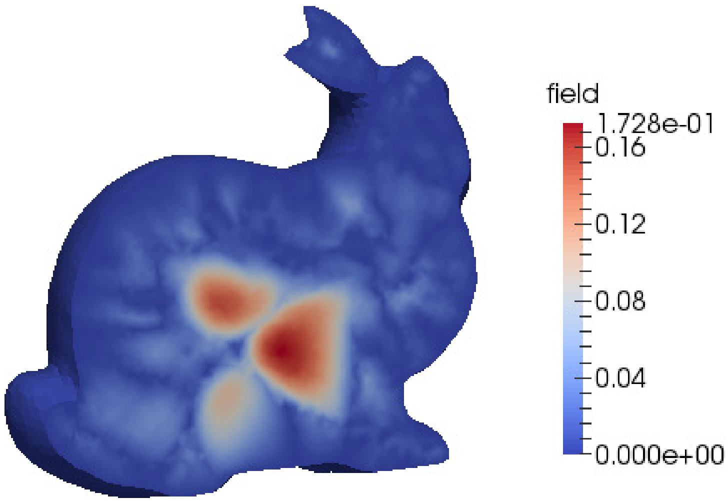
$$E_r(u,\boldsymbol{q},\lambda) = \int_\Omega \left\{ \left(|\boldsymbol{q}|-1\right)^2 + \lambda(\boldsymbol{x})\cdot(\boldsymbol{q}-\nabla u) + r(\boldsymbol{q}-\nabla u)^2 \right\} d\boldsymbol{x} \rightarrow \min$$

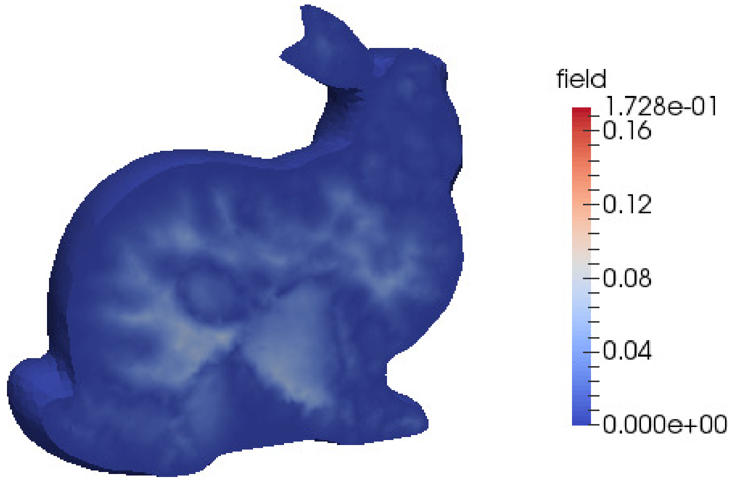
ADMM leads to very similar results

Results for 3D solids

absolute error

$$\left|u(\boldsymbol{x}) - \mathrm{dist}(\boldsymbol{x})\right|$$





distance by $\int_\Omega \left(\left|\nabla u(\boldsymbol{x})\right| - 1\right)^2 d\boldsymbol{x} \to \min$ via splitting scheme

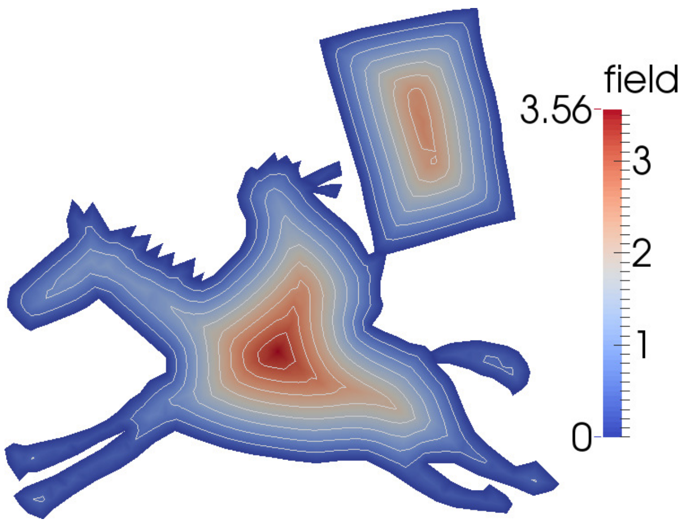
Distance by p-Laplacian (p=8) yields better accuracy

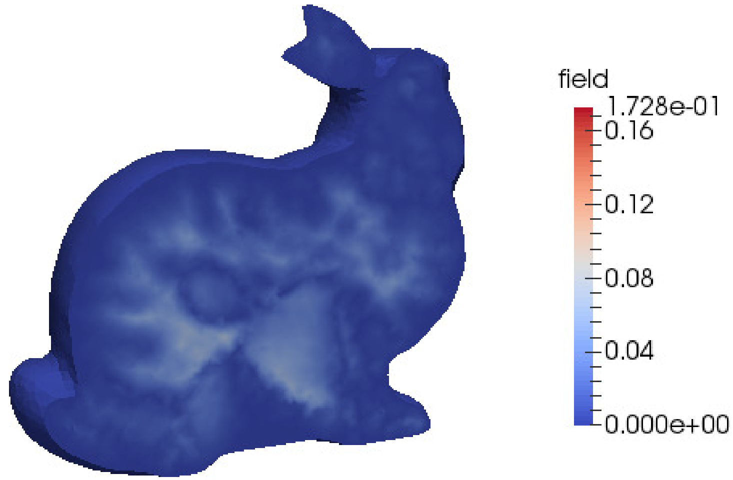
p-Laplacian for distance function estimation

$$\Delta_p u \equiv \mathrm{div}\left(\left|\nabla u\right|^{p-2}\nabla u\right) = -1 \ \text{ in } \ \Omega$$

$$u(\boldsymbol{x}) = 0 \quad \text{on} \ \ \partial\Omega$$

It is known that the solution converges to the distance function as $p \to \infty$





Distance (error) by p-Laplacian (p=8)

Wall distance approximation

The **law of the wall** states that the average velocity of a turbulent flow at a certain point is proportional to the logarithm of the **distance** from that point to the "wall", or the boundary of the fluid region.

$$\Delta\varphi = -1 \quad \text{in} \quad \Omega, \qquad \varphi = 0 \quad \text{on} \quad \partial\Omega$$

$$\psi(\boldsymbol{x}) = \frac{2\varphi}{|\nabla\varphi| + \sqrt{|\nabla\varphi|^2 + 2\varphi}} \quad \Rightarrow \quad \psi = 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\boldsymbol{n}} = 1 \quad \text{on} \quad \partial\Omega$$

Exact for 1-D case

Proposed by P.R.Spalding in 1994

Further developed by P.G.Tucker (Cambridge Uni)

Extending the Spalding-Tucker construction to p-Laplacian:

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -1 \ \text{in} \ \Omega, \qquad u(x) = 0 \ \text{on} \ \partial\Omega$$

$$v(\boldsymbol{x}) = -|\nabla u|^{p-1} + \left[\frac{p}{p-1}u + |\nabla u|^p\right]^{\frac{p-1}{p}} \qquad v = 0$$

$$\partial v / \partial\boldsymbol{n} = 1 \quad \text{on} \quad \partial\Omega$$

Exact for 1-D case

Belyaev-Fayolle 2015

p-Poisson wall distance

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -1 \text{ in } \Omega \qquad u(x) = 0 \text{ on } \partial\Omega$$

$$v(\boldsymbol{x}) = -|\nabla u|^{p-1} + \left[\frac{p}{p-1}u + |\nabla u|^p\right]^{\frac{p-1}{p}} \qquad \begin{cases} v = 0 & \text{on } \partial\Omega \\ \partial v/\partial \boldsymbol{n} = 1 & \text{on } \partial\Omega \end{cases}$$

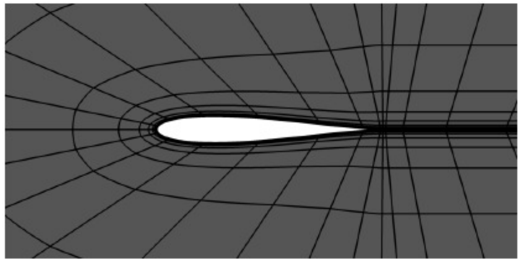
Our p-Poission normalization is used in:

23rd AIAA Computational Fluid Dynamics Conference

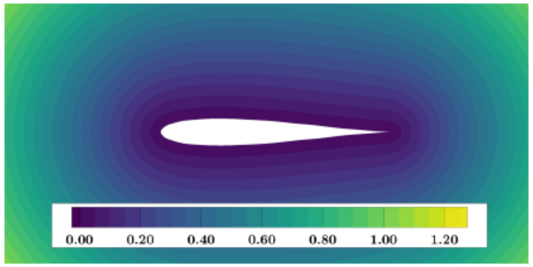
A p-Poisson wall distance approach for turbulence modeling

Nathan A. Wukie* and Paul D. Orkwis†

University of Cincinnati, Cincinnati, Ohio, 45221



(a) Near-body grid.



(b) p-Poisson Distance field: $\tilde{d}$, $p = 6$, $P4$.

Once the solution is obtained, a normalization of the solution to give a better approximation of the distance function was recommended by Belyaev and Fayolle[19] as

$$\tilde{d}(x) = \left[\frac{p}{p-1}u + |\nabla u|^p\right]^{\frac{p-1}{p}} - |\nabla u|^{p-1} \tag{9}$$

p-Laplacian for distance function approximation

$$\Delta_p u \equiv \mathrm{div}\left(\left|\nabla u\right|^{p-2} \nabla u\right) = -1 \ \text{ in } \ \Omega \qquad u(x) = 0 \ \text{ on } \ \partial\Omega$$

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p-Laplace diffusion for distance function estimation, optimal transport approximation, and image enhancement

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A variant of ADMM for numerical solving p-Poisson equation

ADMM for p-Poisson eqaution

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -1 \text{ in } \Omega \qquad u(x) = 0 \text{ on } \partial\Omega$$

$$\int_\Omega \frac{1}{p}|\boldsymbol{\xi}|^p d\boldsymbol{x} - \int_\Omega f\,u\,d\boldsymbol{x} \longrightarrow \min, \quad \text{where} \quad \boldsymbol{\xi} = \nabla u$$

$$\int_\Omega \frac{1}{p}|\boldsymbol{\xi}|^p d\boldsymbol{x} + \frac{r}{2}\int_\Omega (\nabla u - \boldsymbol{\xi})^2 d\boldsymbol{x} - \int_\Omega f\,u\,d\boldsymbol{x} \longrightarrow \min$$

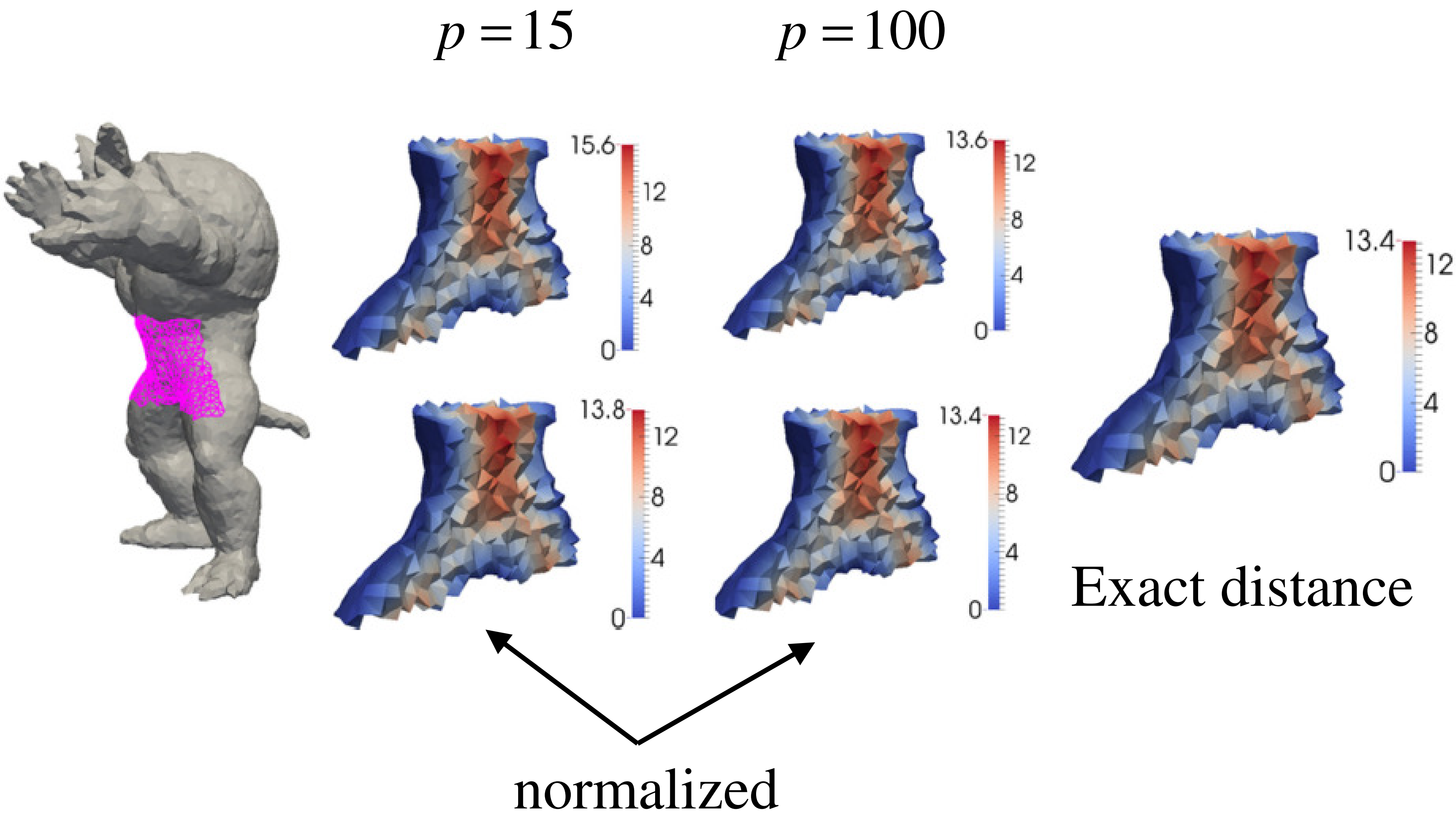
$$-\Delta u = -\operatorname{div}\boldsymbol{\xi} + \frac{1}{r} f \quad \text{in } \Omega$$

$$\frac{1}{p}|\boldsymbol{\xi}|^p + \frac{r}{2}(\nabla u - \boldsymbol{\xi})^2 \rightarrow \min \qquad \boldsymbol{\xi}(\boldsymbol{x}) = c(\boldsymbol{x})\nabla u(\boldsymbol{x})$$

$$\frac{1}{p}c^p|\nabla u|^p + \frac{r}{2}|\nabla u|^2(c-1)^2 \longrightarrow \min$$

$$g(c) \equiv c^{p-1}|\nabla u|^{p-2} + r(c-1) = 0$$

ADMM for p-Poisson eqaution

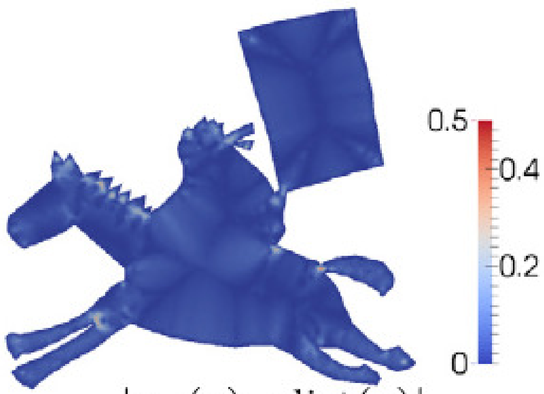


ADMM for p-Poisson equation

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -1 \text{ in } \Omega \qquad u(x) = 0 \text{ on } \partial\Omega$$

$$v(\boldsymbol{x}) = -|\nabla u|^{p-1} + \left[\frac{p}{p-1}u + |\nabla u|^{p}\right]^{\frac{p\text{-}1}{p}}$$

$p = 25$

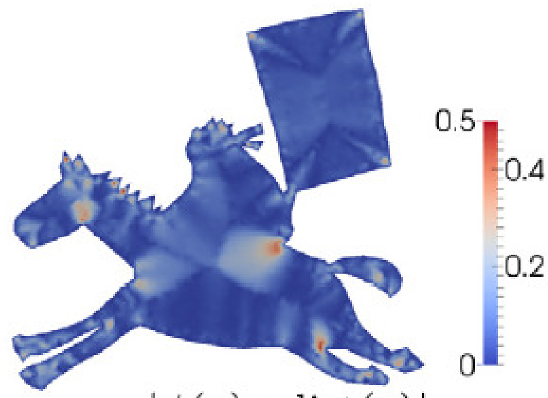


$$\frac{|u_p(\boldsymbol{x}) - \mathrm{dist}(\boldsymbol{x})|}{|\,\mathrm{dist}(\boldsymbol{x})|}$$

p-Laplacian

$$\frac{|v_p(\boldsymbol{x}) - \mathrm{dist}(\boldsymbol{x})|}{|\,\mathrm{dist}(\boldsymbol{x})|}$$

p-Laplacian + normalization



$$\frac{|\phi(\boldsymbol{x}) - \mathrm{dist}(\boldsymbol{x})|}{|\,\mathrm{dist}(\boldsymbol{x})|}$$

geodesics-in-heat

One more way to estimate the distance function

Distance function satisfies

$$\int_{\Omega} \varphi \, d\boldsymbol{x} \to \max, \quad \text{where} \quad \max_{x \in \Omega} |\nabla \varphi| \le 1 \quad \text{and} \quad \varphi(\boldsymbol{x}) = 0 \quad \text{on} \quad \partial\Omega$$

$$F(\varphi) + G(\nabla \varphi) \to \min$$

$$F(\varphi) = -\int_{\Omega} \varphi \, d\boldsymbol{x}$$

$$G(\boldsymbol{q}) = \begin{cases} 0 & \text{if} \quad \|\boldsymbol{q}\|_{L^\infty} \le 1 \\ +\infty & \text{otherwise} \end{cases}$$

ADMM: looking for a saddle point of

$$\int_{\Omega} \left(-\varphi + G(\boldsymbol{q}) + \boldsymbol{\sigma} \cdot (\nabla \varphi - \boldsymbol{q}) + \frac{r}{2} |\nabla \varphi - \boldsymbol{q}|^2 \right) d\boldsymbol{x}$$

One more way to estimate the distance function

$$\text{ADMM:} \qquad \int_\Omega \left(-\varphi + G(\boldsymbol{q}) + \boldsymbol{\sigma} \cdot (\nabla \varphi - \boldsymbol{q}) + \frac{r}{2} \left| \nabla \varphi - \boldsymbol{q} \right|^2 \right) d\boldsymbol{x}$$

$$-r\left(\Delta \varphi_{k+1} - \operatorname{div} \boldsymbol{q}_k\right) = 1 + \operatorname{div} \boldsymbol{\sigma}_k \quad \text{in} \quad \Omega$$

$$\varphi_{k+1}(\boldsymbol{x}) = 0 \quad \text{on} \quad \partial\Omega$$

$$\boldsymbol{q}_{k+1} = P_B\left(\nabla \varphi_{k+1} + \boldsymbol{\sigma}_k / r\right)$$

$$P_B(\boldsymbol{z}) = \begin{cases} \boldsymbol{z} & \text{if} \quad \|\boldsymbol{z}\|_{L^\infty} \leq 1 \\ \boldsymbol{z} / \|\boldsymbol{z}\| & \text{otherwise} \end{cases}$$

$$\boldsymbol{\sigma}_{k+1} = \boldsymbol{\sigma}_k + r\left(\nabla \varphi_{k+1} - \boldsymbol{q}_{k+1}\right)$$

The same approach works for p-Poisson equation

$$\Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = -1 \text{ in } \Omega \qquad u(x) = 0 \text{ on } \partial\Omega$$

$$\left(\int_\Omega \varphi\, d\boldsymbol{x}\right)^p \Big/ \int_\Omega |\nabla\varphi|^p\, d\boldsymbol{x} \to \max \qquad \varphi(\boldsymbol{x}) = 0 \text{ on } \partial\Omega$$

$$\int_\Omega \varphi\, d\boldsymbol{x} \to \max, \text{ where } \|\nabla\varphi\|_{L^p} \le 1 \text{ and } \varphi(\boldsymbol{x}) = 0 \text{ on } \partial\Omega$$

$$F(\varphi) + G_p(\nabla\varphi) \to \min$$

$$F(\varphi) = -\int_\Omega \varphi\, d\boldsymbol{x}$$

$$G(\boldsymbol{q}) = \begin{cases} 0 & \text{if } \|\boldsymbol{q}\|_{L^p} \le 1 \\ +\infty & \text{otherwise} \end{cases}$$

One more way to estimate the distance function

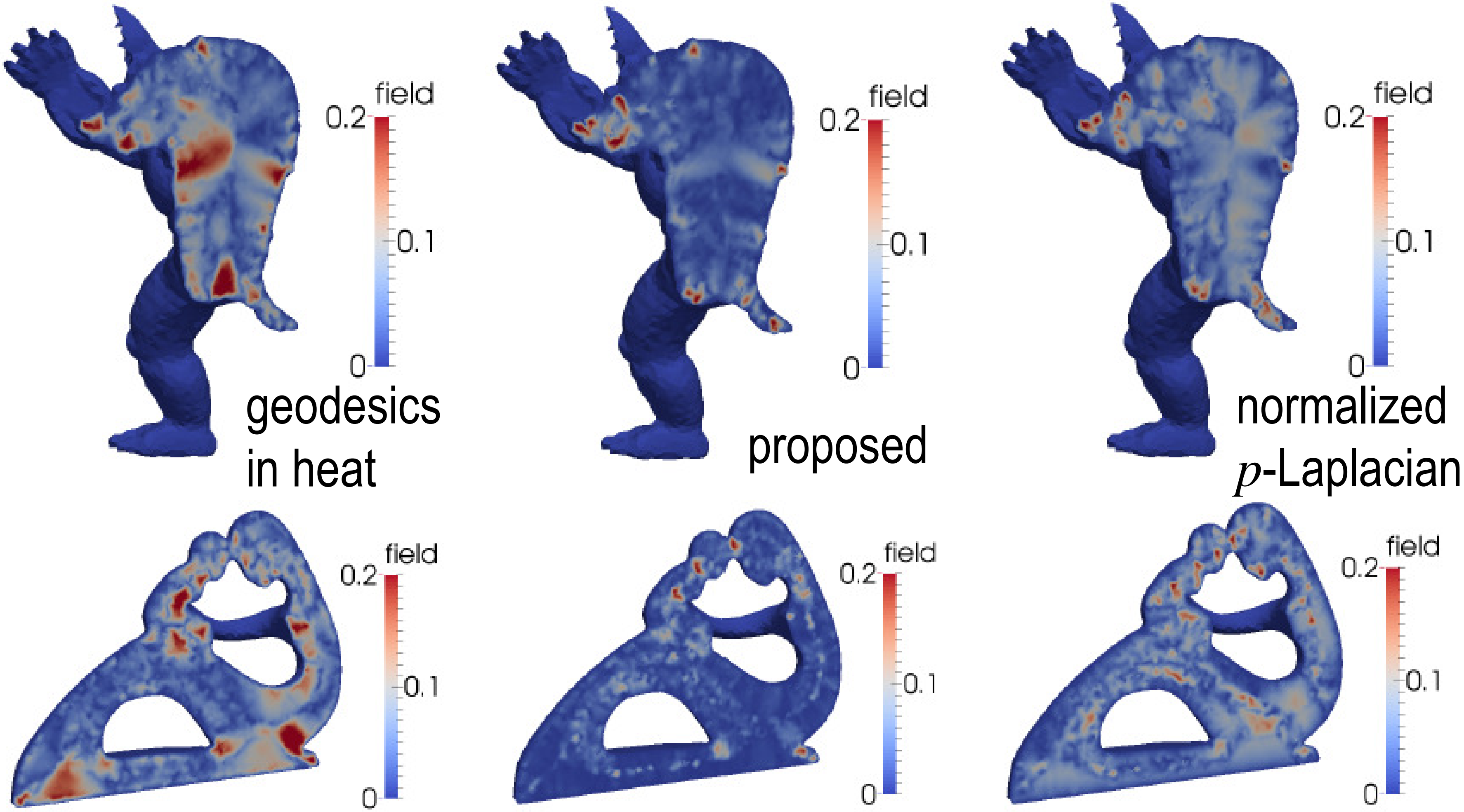


Figure 6: Point-wise relative error w.r.t. the exact distance for the Geodesic-in-Heat distance [8] (left), our variational problem (4) solved numerically by ADMM (middle), and the normalized p-Laplacian distance with $p = 15$ (right).

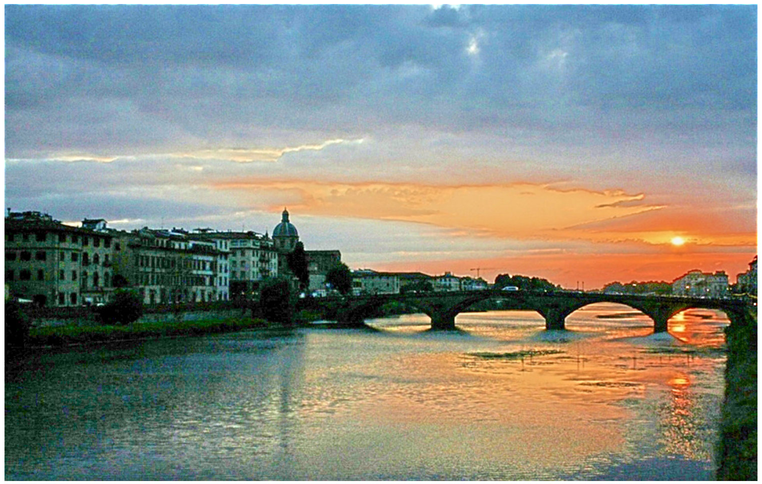
Applications of *p*-Laplacian in Image Processing

$$E(u)=\iint_{\Omega}\left[a(\boldsymbol{x})\left|\nabla u(\boldsymbol{x})\right|^{p(\boldsymbol{x})}+\frac{\lambda}{2}H\left[f(\boldsymbol{x})-u(\boldsymbol{x})\right]^2\right]d\boldsymbol{x}\rightarrow\min$$

$0<p(\boldsymbol{x})<1$ and $a(\boldsymbol{x})$ are constructed from input image $f(\boldsymbol{x})$

$f(\boldsymbol{x})$

$u(\boldsymbol{x})$



The last slide